Some Remarks on Compatibility Classes

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ABSTRACT

In this paper, the notion of partial compatibility relation on a finite set whose non-empty subset is endowed with a compatibility relation is introduced. Moreover, some new properties and results of compatibility classes are presented.

Keywords: *Compatibility relation, Compatibility classes, Maximal compatibility classes, Minimal covering*

1. INTRODUCTION

A relation on a set which is reflexive and symmetric is called a compatibility relation (CR). Compatibility relation is useful in solving a class of incompletely specified phenomena, and in this regard, to the best of the authors' knowledge of this subject, Kurepa [6] seems to be the earliest full-blown mathematical exposition on the study of reflexive symmetric relations and graphs. Compatibility relation has found relevant applications in different fields of knowledge, particularly, in computer science [3, 4, 5, 7, 8, 9, 11, 12].

Notationally, a compatibility relation among the elements of a set is denoted $\approx$. Also, if $\approx$ is a compatibility relation on a set S, then $x, y \subset S$ are called compatible to each other if $x \approx y$.

A family $\{A_1, A_2, \ldots, A_n\}$ of non-empty subsets of S is called a covering of S it $S = \bigcup_{k=1}^{n} A_k$. A cover of a finite set is called minimal if none of its proper subclasses covers S. For some results on covering of S defined by a compatibility relation, please see [13] for further details.

In order to properly describe a compatibility relation, it is important to note that a compatibility relation, not being necessarily transitive, may not define a partition. However, it does define a covering [10]. Essentially, a compatibility relation defined on a finite set decomposes the set into possibly pairwise non-disjoint subclasses, which shall be called compatibility classes (CCs). Consequently, the elements of a CC are pairwise compatible. Some characteristic properties of CCs could be found in [12].

Let S be a finite set and $\approx$ a compatibility relation on S. A subclass $M \subset S$ is called a maximal compatibility class (MCC) if any element of M is compatible to its every other element and no other element of $S - M$ is compatible to all the elements of M. Some results on MCCs of S and an algorithm to compute them could be found in [1], [2], [8] and [12].

In this note, we introduce the notion of partial compatibility relation on a set whose non-empty proper subset is endowed with a compatibility relation; we present some of its characteristics properties and present some results on CCs.

2. PARTIAL COMPATIBILITY RELATION

Usually, in some real life problems which involve compatibility relation among elements or states of a system, it is difficult to implicitly determine the compatibility of all the states of the system. Not very frequently, only the compatibility of some class of states of the system is known. Specifically, in sequential switching machines, such problem is encountered in incompletely specified sequential machine (ISSM) [7] and [8]. This prompts us to study partial compatibility relation.

In [5], a detailed investigation of believability functions induced by generalized partial compatibility relation has been carried out. Our work differs from what is obtained in [5], in that, we study the partial compatibility on a set whose non-empty proper subset is endowed with a compatibility relation. The significance of this is many folds. For example, if we wish to make inference on a super set based upon the compatibility of the members of its non-empty proper subset, partial compatibility relation will play a relevant role. In general, this concept may prove helpful when inference is to be made on system(s) isomorphic to a system upon which a compatibility relation is defined.

Definition 1:

Let S be a finite set and T a non-empty proper subset of S. A compatibility relation on T is called a partial compatibility relation (PCR) on S if the compatibility relation is strictly defined among the elements of T.

Equivalently, we say $\sim$ is a PCR on S if $\sim$ is a CR on T and for the given time t_n, it is not explicitly known if $\sim$ is a CR for the whole of S.

Example 1:

A compatibility relation defined for some but not all elements of S is an example of a PCR. More explicitly, Let $S = \{x_1, x_2, x_3, x_4\}$ and $\alpha = \{a_1, a_2, a_3, a_4, a_5\}$ a set of attributes, such that each attribute is endowed with a discrete nominal scale $A - E$, as represented below.

Table 1: Attributes and states

[1] **Attributes**	[2] x_1	[3] x_2	[4] x_3	[5] x_4	[6
[7] a_1	[8] A	[9] A	[10] A	[11] *	[1
[13] a_2	[14] B	[15] B	[16] C	[17] *	[1
[19] a_3	[20] C	[21] C	[22] D	[23] *	[2
[25] a_4	[26] D	[27] *	[28] D	[29] *	[3
[31] a_5	[32] E	[33] *	[34] *	[35] *	[3

If we define a compatibility relation on S by R as follows:

$R = \{(x_i, x_j) \mid x_i, x_j \in S \wedge x_i \text{ and } x_j \text{ contain some common discrete nominal scale corresponding to attribute } a_i, \text{ for some } i\}$

, where the asterisks (*) represent unspecified discrete nominal scale.

Table 2: compatibility classes table

[37] **CCs**	[38] **Attributes**	[3	[4	[4	[4
[43] $\{x_1, x_2, x_3\}$	[44] a_1	[4	[4	[4	[4
[49] $\{x_1, x_2\}$	[50] a_2, a_3	[5	[5	[5	[5
[55] $\{x_1, x_3\}$	[56] a_4	[5	[5	[5	[6
[61]	[62]	[6	[6	[6	[6
[67]	[68]	[6	[7	[7	[7

Remark 1:

In definition, any element of a finite set that relates only to itself is called an MCC [10, 12]. From the above example, it follows that x_4 is an MCC. But suppose that the discrete nominal scale for x_4 which corresponds to the given attributes is somehow known and coincide with some element x_i, for $i = 1, 2, 3$. What might be appreciated from this knowledge is that x_4 must be part of some MCC, and itself will not be an MCC. Therefore, to avoid this pre-assumption (as a result of this definition of MCC) that x_4 is an MCC, it suffices to restrict the compatibility relation to $T = \{x_1, x_2, x_3\} \subset S$.

Remark 2:

Every PCR on a set S is a CR on S if the CR on $T \subset S$ is a discrete compatibility relation (DCR).
A DCR on a set S is a relation on S which decomposes S into its singleton subsets.

Theorem 1:

Let S be a finite set and T a non-empty proper subset of S. Then any non-discrete compatibility relation defined on T is extendible to S.

Proof:

Let $\approx$ be an arbitrary non-discrete compatibility relation on T. Since $T \neq \emptyset$, there exists some elements $x, y \in T$ such that $x \approx y$. But $T \subset S$ implies that $x \text{ and } y$ are compatible in S. Also, this implies that $S - T \neq \emptyset$. Therefore, any element $z \in (S - T)$ must be either compatible with any $z' \in (S - T)$ or be incompatible with it. At the same time, z is not compatible with any element of T. This discretize some or all elements of $(S - T)$.
In any case, we can find some compatibility relation $\approx'$ which extends $\approx$. Hence the result holds.

Remark 3:

Among the compatibility classes of $T \subset S$ are the proper minimal covers of S.

3. SOME PROPERTIES OF PARTIAL COMPATIBILITY CLASSES

Let S be a finite set endowed with a partial compatibility relation.
The following properties hold:

(i) S has at least two maximal compatibility classes.
(ii) Every minimal covering of S contains at least two disjoint compatibility classes.

Corollary 2:

Let S be a finite set and T a non-empty proper subset of S. Any compatibility relation on T is a compatibility relation on $T \cup \{x\}$, for $x \in (S - T)$.

Proof: Obvious.

4. COMPATIBILITY CLASSES

Graphically, a compatibility class is a complete polygon. A complete polygon is a polygon in which every node is connected to its every other node. A compatibility class may assume the shape of a triangle, a quadrilateral with both diagonals connected and a pentagon with each node connected to every other node is a CC. Note that every subclass of S may not be a CC.

Theorem 3: (Singh and William-west)

For any compatibility class C of S, either C itself is a maximal compatibility class or C is a subclass of some other maximal compatibility class.

Proof:

Let C be a compatibility class of S. If C contains only one element of S, the proof is trivial by definition of MCC. Otherwise, let C contain some $(k > 1)$ elements of S. Suppose $S - C = \emptyset$, the result follows immediately.

Now, let $S - C \neq \emptyset$, then there exists at least one element $x \in S$ such that one of the following hold:

(a) x is not compatible to any element of C
(b) x is compatible to some elements of C, but not all.

In either case, the result holds.

We have added this (theorem 3) result because in [12], it was not proved. However, the gist of theorem 3 is that only compatibility classes could be MCCs, that is, for a candidate maximal compatibility class to be an MCC, it must first be a CC.

Theorem 4:

Let S be a non-empty set. If $\approx$ is a compatibility relation on S, then $\approx$ is a compatibility relation on $S \times S$.

Proof:

The proof follows from the fact that; if $x_i, x_j, x_k, x_l \in S$ are mutually compatible with respect to $\approx$. Then $(x_i, x_j) \approx (x_k, x_l)$ in S. In particular, let $S = \{x\}$ and $\approx$ a compatibility relation on S. Then $\approx$ is a compatibility relation on $S \times S = \{(x, x)\}$. Suppose $S = \{x, y\}$ such that $x \approx y$. Then $\approx$ is a compatibility relation on $S \times S = \{(x, x), (x, y), (y, x), (y, y)\}$, which generates only one compatibility class, namely, $\{x, y\}$. By induction hypothesis, the result is immediate.

Remark 4:

(i) The maximal compatibility classes of $S \times S$ is equivalent to the maximal compatibility classes of S.

(ii) Any compatibility relation defined on a finite set is completely determined by the set of all its maximal compatibility classes. This is a direct consequence of theorem 3.

Proposition 5:

Suppose $\approx$ and $\approx'$ are two compatibility relations on a finite set S, and Γ and Γ' are the sets of all maximal compatibility classes generated by $\approx$ and $\approx'$ respectively. Then $\approx = \approx'$ if and only if $\Gamma = \Gamma'$.

Proof: Easy!

Theorem 6:

Every non-discrete compatibility relation on a set with n-elements generates a family of MCCs Γ with cardinality $n - 1$, at most.

Proof:

Clearly, the cardinality of any family of maximal compatibility classes of an n- set is bounded below and above by 1 and n respectively. But, since the compatibility relation is non-discrete, the result is evident.

5. CONCLUDING REMARKS

This paper has investigated some fundamental concepts of compatibility classes.

Besides presenting some results on compatibility classes of a finite set endowed with a compatibility relation, this paper has introduced the notion of partial compatibility relation and presented some of its properties.

Our future direction is to investigate the applicability of some of these concepts to rough set approximations, multiset multi-attribute model and, artificial neural networks, particularly, when the interactions between neurons are incompletely specified.

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