Boundary control problem for infinite order parabolic system with non-standard functional and time delay

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Abstract—A boundary control problem for infinite order parabolic system with time delay is considered. The performance index has non-standard form. Constraints on controls are imposed. To obtain the optimality conditions for the Dirichlet problem, the generalization of the Dubovitskii-Milyutin theorem was applied. Finally, several mathematical examples for derived optimality conditions are presented.

Keywords: Boundary control problems, Dirichlet problem, Parabolic operators with infinite order and time delay, Dubovitskii-Milyutin Theorem, Conical approximations, Optimality conditions, Weierstrass theorem.

1. Introduction

In recent years, significant emphasis has been given to the study of optimal control of systems governed by parabolic partial differential equations (PPDE) with first boundary conditions or with Cauchy conditions. In these studies, the differential equations are either in general form or in divergence form. It is known that a general class of optimal control problems of systems governed by Itô stochastic differential equations with Markov (fixed) terminal time can be converted into a class of optimal control problems of systems governed by linear second order (PPDE) with first boundary condition (Cauchy condition). Questions concerning necessary conditions for optimality and existence of optimal controls for these problems have been investigated in [30].

In (Refs. [12,15,18,19]), the optimal control problems for systems described by parabolic and hyperbolic operators with infinite order have been discussed. To obtain optimality conditions the arguments of (Ref.[30]) have been applied.

Making use of the Dubovitskii-Milyutin theorem (Ref.[34]), following (Refs. [1-4,23]) Bahaa & Kotarski have obtained necessary and sufficient conditions of optimality for similar systems governed by second order operator with an infinite number of variables and for systems governed by Schrödinger operator. The interest in the study of this class of operators is stimulated by problems in quantum field theory.

In Ref. [22], Kotarski considered an optimization problem for a parabolic system and obtained necessary and sufficient conditions for optimality by applying the classical Dubovitskii-Milyutin Theorem (Ref. [20]). The performance index was more general than the quadratic one and had an integral form. The set representing the constraints on controls was assumed to have a nonempty interior. This assumption can be easily removed if we apply the generalization of the Dubovitskii-Milyutin Theorem (Ref.[34]), instead of the classical one (Ref. [20]) (as the approximation of the set of controls, the regular tangent cone is used instead of the regular admissible cone).

Partial differential equations with delay are arising in a wide area of applied mathematics see for example [23,24,27]. In these papers, a distributed control problem for parabolic and hyperbolic operator with an infinite number of variables and operators of infinite order with time delay is considered.

In this paper the application of the generalized Dubovitskii-Milyutin Theorem will be demonstrated on an optimization problem with delay for a system described by a parabolic operator with infinite order and with Dirichlet boundary conditions. The performance index has an integral form. Constraints on controls are imposed. To obtain optimality conditions for the Dirichlet problem, the generalization of the Dubovitskii-Milyutin Theorem given by Walczak in Refs.[33,34], was applied.

This paper is organized as follows. In section 2, we introduce some functional spaces with infinite order. In section 3, we define a parabolic equation with infinite order and time delay. In section 4, we formulate the optimal control problem and we introduce the main results of this paper. The end of this section contains several examples to illustrate the main result of the paper.

II. INFINITE-ORDER FUNCTIONAL SPACES AND GENERALIZED DUBOVITSKII-MILYUTIN THEOREM

The object of this section is to give the definition of some functional spaces of infinite order, and the chains of the constructed spaces which will be used later and also we state the generalized Dubovitskii Milyutin theorem (Refs.[5],[22]). We firstly define the Sobolev space \( W^\infty_2(a,2)(\mathbb{R}^n) \) (which we will denote by \( W^\infty(a,2) \)) of infinite order of functions \( \phi(x) : \mathbb{R}^n \to C^1, n \geq 1 \), as follows,

\[
W^\infty(a,2) = \left\{ \phi(x) \in C^\infty(\mathbb{R}^n) : \sum_{|\alpha| = 0}^\infty a_\alpha ||D^\alpha \phi||_2^2 < \infty \right\}
\]

where \( a_\alpha \geq 0 \) is a numerical sequence and \( ||\cdot||_2 \) is the canonical norm in the space \( L^2(\mathbb{R}^n) \) (all functions are assumed to be real valued), and

\[
D^\alpha = \frac{\partial^{|\alpha|}}{(\partial x_1)^{\alpha_1} \cdots (\partial x_n)^{\alpha_n}}.
\]
where \( \alpha = (\alpha_1, ..., \alpha_n) \) is a multi-index for differentiation, 
\[ |\alpha| = \sum_{i=1}^{n} \alpha_i. \]

The space \( W^\infty \{a_2, a\} \) is non trivial if and only if the characteristic function of this space
\[ a(\xi) \equiv \sum_{|\alpha|=0}^{\infty} a_\alpha \xi^{2\alpha}, \quad \xi \in \mathbb{R}^n, \]
is analytic in some neighborhood of the point \( \xi = 0 \). We note, however, that if the domain of convergence of \( a(\xi) \) is not the whole space \( \mathbb{R}^n \), then the space \( W^\infty \{a_2, a\} \) is not dense in \( L^2(\mathbb{R}^n) \). This "negative" fact manifests itself in the theory of differential equations in that it does not permit going from generalized solutions to classical solutions. In order to avoid this, we assume in what follows that the characteristic function \( a(\xi) \) is an entire function.

The space \( W^\infty \{a_2, a\} \) is defined as the formal conjugate space to the space \( W^\infty \{a_2, a\} \), namely:
\[ W^\infty \{a_2, a\} = \left\{ \psi(x) : \psi(x) = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha \psi_\alpha(x) \right\}, \]
where \( \psi_\alpha \in L^2(\mathbb{R}^n) \) and \( \sum_{|\alpha|=0}^{\infty} a_\alpha ||\psi_\alpha||^2_2 < \infty. \)

The duality pairing of the spaces \( W^\infty \{a_2, a\} \) and \( W^\infty \{a_2, a\} \) is postulated by the formula
\[ \langle \phi, \psi \rangle = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_{\mathbb{R}^n} \psi_\alpha(x) D^\alpha \phi(x) dx, \]
where
\[ \phi \in W^\infty \{a_2, a\}, \quad \psi \in W^\infty \{a_2, a\}. \]

From above, \( W^\infty \{a_2, a\} \) is everywhere dense in \( L^2(\mathbb{R}^n) \) with topological inclusions and \( W^\infty \{a_2, a\} \) denotes the topological dual space with respect to \( L^2(\mathbb{R}^n) \), so we have the following chain:
\[ W^\infty \{a_2, a\} \subseteq L^2(\mathbb{R}^n) \subseteq W^\infty \{a_2, a\}. \]

Analogous to the above chain we have:
\[ W^\infty_0 \{a_2, a\} \subseteq L^2(\mathbb{R}^n) \subseteq W^\infty_0 \{a_2, a\}, \]
where \( W^\infty_0 \{a_2, a\} \) is the set of all functions of \( W^\infty \{a_2, a\} \) which vanish on the boundary \( \Gamma \) of \( \mathbb{R}^n \) (\( \Gamma \) is meant to be the boundary of the support of the measure \( dx \)), i.e.,
\[ W^\infty_0 \{a_2, a\} = \left\{ \phi \in W^\infty_0(\mathbb{R}^n) : ||\phi||^2 = \sum_{|\alpha|=0}^{\infty} a_\alpha ||D^\alpha \phi||^2_2 < \infty, D^\alpha \phi |\Gamma = 0, |\omega| = |\alpha - 1| \right\}. \]

We now introduce \( L^2(0, T; L^2(\mathbb{R}^n)) \) which we will denote by \( L^2(Q) \), where \( Q = \mathbb{R}^n \times [0, T] \), denotes the space of measurable functions \( t \rightarrow \phi(t) \) such that
\[ ||\phi||_{L^2(Q)} = \left( \int_0^T ||\phi(t)||^2_2 dt \right)^{\frac{1}{2}} < \infty, \]
endowed with the scalar product \( (f, g) = \int_0^T (f(t), g(t))_{L^2(\mathbb{R}^n)} dt \), \( L^2(Q) \) is a Hilbert space. In the same manner we define the spaces \( L^2(0, T; W^\infty \{a_2, a\}) \), \( L^2(0, T; W^\infty_0 \{a_2, a\}) \) and \( L^2(0, T; W^{-\infty} \{a_2, a\}) \) as its formal conjugate resp.

Also, we have the following chains:
\[ L^2(0, T; W^\infty \{a_2, a\}) \subseteq L^2(Q) \subseteq L^2(0, T; W^\infty \{a_2, a\}), \]
\[ L^2(0, T; W^\infty_0 \{a_2, a\}) \subseteq L^2(Q) \subseteq L^2(0, T; W^{-\infty} \{a_2, a\}). \]

Let us introduce the space
\[ W(-\tau_0, T) := \left\{ y : y \in L^2(0, T; W^\infty_0 \{a_2, a\}) \right\}, \]
in which a solution of a parabolic equation with infinite order will be contained.

Let \( C_i, i = 1, 2, ..., n \) be a system of cones and \( B_M \) be a ball with center 0 and radius \( M > 0 \) in the space \( X \).

A. Definition 2.1 [22]

The cones \( C_i, i = 1, 2, ..., n \) are of the same sense if \( \forall M > 0, \exists M_1, ..., M_n > 0 \) so that \( \forall x \in B_M \cap \sum_{i=1}^{n} C_i, x = \sum_{i=1}^{n} x_i, x_i \in C_i, i = 1, 2, ..., n \), we have \( x_i \in B_{M_i}, C_i \) and \( \sum_{i=1}^{n} C_i, i = 1, 2, ..., n \) (equivalently the inequality \( ||x|| \leq M \) implies the inequalities \( ||x_i|| \leq M_i, i = 1, 2, ..., n \).

B. Definition 2.2 [22]

The cones \( C_i, i = 1, 2, ..., n \) are of the opposite sense if \( \exists (x_1, ..., x_n) \neq (0, 0, 0), x_i \in C_i, i = 1, 2, ..., n \) so that
\[ 0 = \sum_{i=1}^{n} x_i. \]

C. Remarks 2.1 [22]

1. From definitions 2.1 and 2.2 it follows that the set of cones of the same sense is disjoint with the set of cones of the opposite sense. If a certain subsystem of cones is of the opposite sense, then the whole system is also of the opposite sense.

2. In finite dimensional spaces only the cones of the two types mentioned above [34] may exist while in arbitrary infinite dimensional normed spaces the situation is more complicated.

In [34] the condition under which a system of cones is of the same sense are given.

We define the Problem (P): find \( x_0 \in Q \) such that
\[ \min_{x \in Q \cap U(x^0)} I(x) = I(x^0), \]
where \( Q = \bigcap_{k=1}^{n} Q_k \) and \( U(x^0) \) is some neighbourhood of \( x^0 \).

If we define equality constraints in the operator form:
\[ Q_k := \{ x \in X : F_k(x) = 0 \} \]
where \( F_k : X \rightarrow Y_k \) are given operators, \( Y_k \) are Banach spaces, \( k = p + 1, ..., n \), then we obtain Problem (P1) instead of Problem (P).

We state now the Generalized Dubovitskii-Milyutin Theorem:
D. Theorem (Generalized Dubovitskii-Milyutin Theorem)

We assume for problem (P) that:
(i) the cones $K_i, i = 1, ..., s, D_j, j = 1, ..., s, C_k, k = 1, ..., p,$ are open and convex,
(ii) the cones $C_k, k = p + 1, ..., n$ are convex and closed,
(iii) the cone $\mathcal{C} = \bigcap_{k=p+1}^{n} C_k$ is contained in the cone tangent

to the set $\bigcap_{k=p+1}^{n} Q_k$,
(iv) the cones $C_k, k = p + 1, ..., n$ are either of the same sense or of the opposite sense,
(v) $x^0 \in Q$ is a local optimum for problem (P),

then the following "s" equations (the so-called Euler-Lagrange equations) must hold:

$$f_i + \sum_{j=1, j \neq i}^{s} f_{(i)}^j + \sum_{k=1}^{n} \varphi_{k}^{(i)} = 0, \quad i = 1, 2, ..., s,$$

where $f_i \in K_i^*, f_{(i)}^j \in D_j^2, j = 1, ..., s, j \neq i, \varphi_{k}^{(i)} \in C_k^*, k = 1, ..., n$, with not all functionals equal to zero simultaneously.

III. INFINITE-ORDER PARABOLIC EQUATION WITH TIME DELAY

In Ref.[14], which is a review article for previous results that earlier obtained by I. M. Gali, H. A. El-Saify and S. A. El-Zahaby [16-19], the results obtained there are for the case of operators with an infinite number of variables which are elliptic, parabolic, hyperbolic or well-posed in the sense of Petrovsky.

Subsequently, J. L. Lions (see [30]) suggested a problem related to this result but in different direction by taking the case of operators of infinite order with finite dimension in the form:

$$(A\Phi)(x) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} \Phi(x), \quad x \in \mathbb{R}^n.
$$

The operator $A$ is a bounded self-adjoint elliptic operator with infinite order mapping $W^{2\infty}_0(\mathbb{R}^n)$ onto $W^{2\infty}_0(\mathbb{R}^n)$. Let $\tau_0 > 0$ be a given number representing the time delay. We consider the following evolution equation with a solution in the space $W(\tau_0, T)$ (Refs.[23,24,27]):

$$\frac{\partial y(x,t)}{\partial t} + Ay(x,t) + y(x,t - \tau_0) = 0, \quad x \in \mathbb{R}^n, \quad t \in (0, T),$$

$$y(x, t) = g(x, t), \quad x \in \mathbb{R}^n, \quad t \in (-\tau_0, 0),$$

$$y(x, 0) = y_p(x), \quad x \in \mathbb{R}^n,$$

$$D^{\omega} y(x,t) = f, \quad x \in \Gamma, \quad t \in (-\tau_0, T), \quad |\omega| = 0, 1, 2, ..., |\omega| \leq \alpha - 1, \quad |\alpha| > 0,$$

where

$$f \in L^2(0, T; W^{\infty}_0(\Gamma)), \quad g \in W(\tau_0, T), \quad y_p \in L^2(\mathbb{R}^n)$$

are given functions.

For each $t \in (0, T]$, we define the following bilinear form on $W_0^{\infty}(\mathbb{R}^n)$:

$$\pi(t; \phi, \psi) = (A\phi, \psi)_{L^2(\mathbb{R}^n)}, \quad \phi, \psi \in W_0^{\infty}(\mathbb{R}^n).$$

Then

$$\pi(t; \phi, \psi) = \left(\int_{\mathbb{R}^n} \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} \phi(x) \psi(x) \right)_{L^2(\mathbb{R}^n)}$$

The bilinear form (3.5) is coercive on $W_0^{\infty}(\mathbb{R}^n)$ that is, there exists $\eta \in \mathbb{R}$, such that:

$$\pi(t; \phi, \phi) = \eta \|\phi\|_{W_0^{\infty}(\mathbb{R}^n)}^2, \quad \eta > 0.$$

Indeed, it is well known that the ellipticity of $A$ is sufficient for the coerciveness of $\pi(t; \phi, \psi)$ on $W_0^{\infty}(\mathbb{R}^n)$ [30]. Then

$$\pi(t; \phi, \psi) = \left(\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha} \phi(x), \phi(x) \right)_{L^2(\mathbb{R}^n)}$$

Also we have the symmetric condition of $\pi$ i.e. $\forall \phi, \psi \in W_0^{\infty}(\mathbb{R}^n)$ the function $t \rightarrow \pi(t; \phi, \psi)$ is continuously differentiable in $[0, T]$; and

$$\pi(t; \phi, \psi) = \pi(t; \phi, \psi). \quad (3.7)$$

Note. The operator $\frac{\partial}{\partial t} + A$ is an infinite order parabolic operator which maps $L^2(0, T; W_0^{\infty}(\mathbb{R}^n))$ onto $L^2(0, T; W_0^{\infty}(\mathbb{R}^n))$. Equations (3.1)-(3.4) can be presented in a more convenient form, after defining the linear and bounded operators $M$ and $N$ as follows:

$$M y(t) = \begin{cases} y(t - \tau_0), & t > 0, \\ 0, & t < 0 \end{cases}$$

$$N f(t) = \begin{cases} f(t), & t > 0, \\ \frac{\partial}{\partial t} + A g, & t < 0 \end{cases}$$

$$y(x, 0) = g(x, 0).$$

Then, Eqs. (3.1)-(3.4) take the form

$$\frac{\partial y(x,t)}{\partial t} + Ay(x,t) + My(x,t) = 0, \quad x \in \mathbb{R}^n, \quad t \in (-\tau_0, T),$$

$$y(x, 0) = g(x, 0), \quad x \in \mathbb{R}^n,$$

$$D^{\omega} y(x,t) = N f, \quad x \in \Gamma, \quad t \in (-\tau_0, T).$$

It is easy to notice that, in the interval $(0, T)$, Eq. (3.11) is equivalent to (3.1). But in the interval $(-\tau_0, 0)$ from Eqs. (3.11)-(3.13) we get

$$\frac{\partial y(x,t)}{\partial t} + Ay(x,t) = 0, \quad x \in \mathbb{R}^n, \quad t \in (-\tau_0, 0).$$
\[ y(x, 0) = g(x, 0), \quad x \in \mathbb{R}^n, \quad (3.15) \]
\[ D^\omega y(x, t) = \frac{\partial g(x, t)}{\partial t} + Ay(x, t), \quad x \in \Gamma, \quad t \in (-\tau_0, 0). \quad (3.16) \]

On the basis of the theorem on the existence and the uniqueness of the solution of the parabolic equation (Theorem 3.1, Ref.[22]), we can see that \( y \equiv g \) in \((-\tau_0, 0)\).

The existence and uniqueness of the solution of Eqs. (3.1)-(3.4) can be proved using a constructive method, i.e., firstly solving the Eqs. (3.1)-(3.4) in the subinterval \((0, \tau_0)\) and then in the subinterval \((\tau_0, 2\tau_0)\) one etc., until one covers the whole interval \((0, T)\); thus, the solution in the previous step is the initial condition for the next one. Really, in every subinterval the solution of Eqs. (3.1)-(3.4) exists and is unique (Theorem 3.1, Ref.[22]). If the right-hand sides of Eqs. (3.1)-(3.4) are equal to zero, then there exists only trivial solution of Eqs. (3.1)-(3.4), i.e., \( y \equiv 0 \). A more general case (e.g., when \( \tau_0 \) is a function of the time \( t \)) can be treated analogously as in Ref. [33].

**IV. PROBLEM FORMULATION AND OPTIMIZATION THEOREM**

This section is devoted to consider the following boundary optimization problem:

\[ \frac{\partial y(x, u; t)}{\partial t} + Ay(x, u; t) + g(x, u; t - \tau_0) = 0, \quad (4.1) \]
\[ x \in \mathbb{R}^n, \quad t \in (0, T), \]
\[ y(x, u; t) = g(x, u; t), \quad x \in \mathbb{R}^n, \quad t \in (-\tau_0, 0), \quad (4.2) \]
\[ y(x, u; 0) = y_p(x, u), \quad x \in \mathbb{R}^n, \quad (4.3) \]
\[ D^\omega y(x, u; t) = u, \quad x \in \Gamma, \quad t \in (-\tau_0, T). \quad (4.4) \]

Let us denote by \( u \in U := L^2(0, T; L^2(\Sigma)) = L^2(\Sigma), \) the space of controls and by \( y \in Y := L^2(-\tau_0, T; W_0^{2,1}(\mathbb{R}^n)) \) the space of states, \( y_p \) is a given element in \( L^2(\mathbb{R}^n) \), \( g \) is a given element in \( W(-\tau_0, 0) \). The control time \( T \) is fixed in our problem (Refs.[23,24,27]).

The performance functional is given by the following non-standard functional:

\[ I(y, u) = \left( I_1(y, u) \right)^{\alpha_1} \times \left( I_2(y, u) \right)^{\alpha_2} \]
\[ = \left( \int_\Sigma F_1(x, t, y, u)d\Gamma dt \right)^{\alpha_1} \times \left( \int_\Sigma F_2(x, t, y, u)d\Gamma dt \right)^{\alpha_2} \quad \min, \quad (4.5) \]

where \( \alpha_1, \alpha_2 \in [0, 1], \) and \( F_1: \Gamma \times (0, T) \times \mathbb{R}^1 \times \mathbb{R}^1 \rightarrow \mathbb{R}^1, \)
\( i = 1, 2 \) satisfies the following conditions:

\( A_1 \) \( F_1(x, t, y, u) \) is continuous with respect to \((x, t, y, u), \)
\( A_2 \) there exists \( F_{iy}(x, t, y, u), F_{iu}(x, t, y, u) \) which are continuous with respect to \((x, t, y, u), \)
\( A_3 \) \( 0 < \int_\Sigma F_1(x, t, y, u)d\Gamma dt < +\infty. \)

\((y^0, u^0)\) denote the optimal state and the optimal control respectively.

**Control constraints.**

We assume the following constraints on controls:

\[ u \in U_{ad} \] is a closed, convex subset in the space \( L^2(\Sigma), \quad (4.6) \]

We formulate the necessary conditions of optimality for the problem (4.1)-(4.6) in the following optimization theorem.

**A. OPTIMIZATION THEOREM**

Let \( (y^0, u^0) \) be the optimal solution to the optimal control problem (4.1)-(4.6). Then with the assumptions mentioned above, there is \( p \in W(-\tau_0, T) \), satisfying (in the weak sense) the adjoint equation given below and the following system of partial differential equations and inequalities must be satisfied:

**State equations:**

\[ \frac{\partial y^0(x, u; t)}{\partial t} + Ay^0(x, u; t) + y^0(x, u; t - \tau_0) = 0, \quad x \in \mathbb{R}^n, \quad t \in (0, T), \]
\[ y^0(x, t) = g(x, t), \quad x \in \mathbb{R}^n, \quad t \in (-\tau_0, 0), \]
\[ y^0(x, 0) = y_p(x), \quad x \in \mathbb{R}^n, \quad (4.9) \]
\[ D^\omega y(x, u; t) = u^0, \quad x \in \Gamma, \quad t \in (-\tau_0, T). \quad (4.10) \]

**Adjoint equations:**

\[ -\frac{\partial p(x, u; t)}{\partial t} + A^*p(x, u; t) + p(x, u; t + \tau_0) = (N_1F_y, x \in \mathbb{R}^n, \quad t \in (-\tau_0, T - \tau_0), \]
\[ -\frac{\partial p(x, u; t)}{\partial t} + A^*p(x, u; t) = N_1F_y, \quad x \in \mathbb{R}^n, \quad t \in (T - \tau_0, T), \]
\[ p(x, u; T) = 0, \quad x \in \mathbb{R}^n, \quad (4.12) \]
\[ D^\omega p(x, u; t) = 0, \quad x \in \Gamma, \quad t \in (-\tau_0, T). \quad (4.14) \]

**Maximum conditions:**

\[ \int_\Sigma \frac{\partial p(x, u; t)}{\partial n} + \alpha_1 F_{1u} + \alpha_2 F_{2u}(u - u^0)d\Gamma \geq 0 \]
\[ \forall u \in U_{ad}, \quad (4.15) \]

where
\[ \lambda = \frac{\int_\Sigma F_1(x, t, y^0, u^0)d\Gamma dt}{\int_\Sigma F_2(x, t, y^0, u^0)d\Gamma dt}, \]
\( F_{iy}, F_{iu}, i = 1, 2 \) are the Fréchet derivatives of \( F \) with respect to \( y, u \) respectively at the point \((y^0, u^0)\), \( \frac{\partial y}{\partial n} \) is the co-normal derivatives with respect to \( A \) in the direction \( u \), and

\[ N_1F_y = F_y, \quad t > 0, \quad t < 0. \]

**Proof.**
Firstly it is easy to notice that the functional \(lnI\) increases monotonically with \(I\) and the minima of the functional (4.5) and
\[
G(y, u) = lnI(y, u) = \sum_{i=1}^{2} \alpha_i ln \int_{\Sigma} F_i(x, t, y, u) dt
\]
must be achieved simultaneously at the same point. Thus we can replace the functional \(I(y, u)\) by \(G(y, u)\) and for such a new problem we can formulate necessary conditions of optimality by applying the generalized Dubovitskii-Milyutin Theorem.

Let us denote by \(Q_1, Q_2\) the sets in the space \(E := Y \times U\) as follows
\[
\begin{align*}
Q_1 & := \left\{ (y, u) \in E; \quad \begin{cases} 
\frac{\partial y}{\partial t} + Ay + My = 0, & x \in \mathbb{R}^n, \quad t \in (-\tau_0, T), \\
y(x, 0) = y_p(x), & x \in \mathbb{R}^n, \\
D^0 y(x, t) = Nu, & x \in \Gamma, \quad t \in (-\tau_0, T)
\end{cases}
\right\}, \\
Q_2 & := \left\{ (y, u) \in E; \quad y \in Y, u \in U_{ad}\right\}.
\end{align*}
\]
Thus the optimization problem may be formulated in such a form
\[
G(y, u) \rightarrow \min \quad \text{subject to} \quad (y, u) \in Q_1 \cap Q_2.
\]

We approximate the sets \(Q_1, Q_2\) by the regular tangent cones (\(RTC\)), and the performance index by the regular cone of decrease (\(RFC\)).

The tangent cone to the set \(Q_1\) at \((y^0, u^0)\) has the form
\[
RTC(Q_1, (y^0, u^0)) = \left\{ (\overline{y}, \overline{u}) \in E; \quad P'(y^0, u^0)(\overline{y}, \overline{u}) = 0 \right\}
\]
where \(P'(y^0, u^0)(\overline{y}, \overline{u})\) is the Fréchet differential of the operator
\[
P(y, u) := \left( \frac{\partial y}{\partial t} + Ay + My, y(x, 0) - y_p(x), D^0 y(x, t) - Nu \right)
\]
into the space
\[
W := L^2(-\tau_0, T; W_0^{\infty}(\mathbb{R}^n)) \times L^2(\Sigma)
\]
and
\[
Z := L^2(-\tau_0, T; W_0^{\infty}(\mathbb{R}^n)) \times L^2(\mathbb{R}^n).
\]
Applying theorem on the existence of the solution to the equation (4.1)-(4.3), it is easy to prove that \(P'(y^0, u^0)\) is the mapping from the space \(W\) onto \(Z\) as required in the Lusternik Theorem (Theorem 9.1 in [20]).

The tangent cone \(RTC(Q_2, (y^0, u^0))\) to the set \(Q_2\) at \((y^0, u^0)\) has the form \(Y \times RTC(U_{ad}, u^0)\), where \(RTC(U_{ad}, u^0)\) is the tangent cone to the set \(U_{ad}\) at the point \(u^0\). It is known that the tangent cones are closed [34]. Then we can show that:
\[
RTC(Q_1 \cap Q_2, (y^0, u^0)) = RTC(Q_1, (y^0, u^0)) \cap RTC(Q_2, (y^0, u^0)).
\]

Following [33] it is easy to show that
\[
RTC(Q_1 \cap Q_2, z^0) = RTC(Q_1, z^0) \cap RTC(Q_2, z^0).
\]
We only need to show the inclusion ”\(\supset\)”, because we always have ”\(\subset\)” [28].

It can be easily checked that in the neighborhood \(V_0\) of the point \((y^0, u^0)\) the operator \(P\) satisfies the assumptions of the implicit function theorem [21]. Consequently the set \(Q_1\) can be represented in the neighborhood \(V_0\) in the form
\[
\left\{ (y, u) \in E; \quad y = \varphi(u) \right\}, \tag{4.16}
\]
where \(\varphi : L^2(\Sigma) \rightarrow L^2(0, T; W_0^{\infty}(\alpha, 2)) \times L^2(\Sigma)\) is the operator of class \(C^1\) satisfying the condition \(P(\varphi(u), u) = 0\) for \(u\) such as \((\varphi(u), u) \in V_0\).

From this we get
\[
RTC(Q_1, z^0) = \left\{ \overline{z} \in E; \overline{y} = \varphi(u^0)\overline{y} \right\}. \tag{4.17}
\]
Let \((\overline{y}, \overline{u})\) be any element of the set \(RTC(Q_1, z^0) \cap RTC(Q_2, z^0)\).
From the definition of the tangent cone we know that there exists the operator $r^2_u := \mathbb{R}^1 \rightarrow U$ such as $\frac{r^2_u(\epsilon)}{\epsilon} \rightarrow 0$ with $\epsilon \rightarrow 0^+$ and

$$
(y^0, u^0) + \epsilon(\bar{y}, \bar{u}) + (r^2_u(\epsilon)) \in Q_2
$$

(4.18)

for a sufficiently small $\epsilon$ and with any $r^2_u(\epsilon)$.

From (4.16) results that for sufficiently small $\epsilon$, we get

$$
\left( \varphi(u^0 + \epsilon \bar{u} + r^2_u(\epsilon)), u^0 + \epsilon \bar{u} + r^2_u(\epsilon) \right) \in Q_1.
$$

Since $\varphi$ is a differentiable operator therefore

$$
\varphi(u^0 + \epsilon \bar{u} + r^2_u(\epsilon)) = \varphi(u^0) + \epsilon \varphi_u(u^0) \bar{u} + r^1_u(\epsilon)
$$

for some $r^1_u(\epsilon)$ such as $r^2_u(\epsilon) \rightarrow 0$ with $\epsilon \rightarrow 0^+$.

Taking into account (4.16) and (4.17), we can get

$$
(y^0, u^0) + \epsilon(\bar{y}, \bar{u}) + (r^1_u(\epsilon), r^2_u(\epsilon)) \in Q_1.
$$

(4.19)

If in (4.18) we have $r^2_u(\epsilon) = r^1_u(\epsilon)$, then it results from (4.18) and (4.19) that $(\bar{y}, \bar{u})$ is an element of the cone tangent to the set $Q_1 \cap Q_2$ at $z^0$. It completes the proof of the inclusion $^\sim \subset ^\sim$. Further applying Theorem 3.3 [34] we can prove that the adjoint cones $[RTC(Q_1, z^0)]^\sim$ and $[RTC(Q_2, z^0)]^\sim$ are of the same sense.

According to Theorem 7.5 [20] the regular cone of decrease for the performance functional is given by

$$
RFC(G, (y^0, u^0)) = \left\{ (\bar{y}, \bar{u}) \in E; G'(y^0, u^0)(\bar{y}, \bar{u}) < 0 \right\},
$$

where $G'(y^0, u^0)(\bar{y}, \bar{u})$ is the Fréchet differential of the performance functional $G(y, u)$. With the rule on the differentiation of composed functions and assumptions (A1), (A2), (A3) such differential exists (the existence of Fréchet differentials of $f \in \mathcal{C}^1$).

$$
RFC(G, (y^0, u^0)) = \left\{ (\bar{y}, \bar{u}) \in E; \alpha_1 \int_{\Sigma} (F_{1y} \bar{y} + F_{1u} \bar{u}) d\Gamma dt + \alpha_2 \int_{\Sigma} (F_{2y} \bar{y} + F_{2u} \bar{u}) d\Gamma dt < 0 \right\}
$$

Hence

$$
RFC(G, (y^0, u^0)) = \left\{ (\bar{y}, \bar{u}) \in E; \alpha_1 \int_{\Sigma} (F_{1y} \bar{y} + F_{1u} \bar{u}) d\Gamma dt + \alpha_2 \int_{\Sigma} (F_{2y} \bar{y} + F_{2u} \bar{u}) d\Gamma dt < 0 \right\}
$$

(4.20)

If $RFC(G, (y^0, u^0)) \neq 0$, then the adjoint cone consists of the elements of the form (Theorem 10.2 [20])

$$
f_3(\bar{y}, \bar{u}) = -\lambda_0 \left[ \alpha_1 \int_{\Sigma} (F_{1y} \bar{y} + F_{1u} \bar{u}) d\Gamma dt + \alpha_2 \int_{\Sigma} (F_{2y} \bar{y} + F_{2u} \bar{u}) d\Gamma dt \right]
$$

(4.20)

where $\lambda_0 \geq 0$.

Since $RTC(Q_1, (y^0, u^0))$ is a subspace of $E$, then the functionals belonging to $[RTC(Q_1, (y^0, u^0))]^\sim$ have the form (Theorem 10.1 [20])

$$
f_1(\bar{y}, \bar{u}) = 0 \quad \forall (\bar{y}, \bar{u}) \in RTC(Q_1, (y^0, u^0)).
$$

The functionals $f_2(\bar{y}, \bar{u}) \in [RTC(Q_2, (y^0, u^0))]^\sim$ can be expressed as follows

$$
f_2(\bar{y}, \bar{u}) = f^2_2(\bar{y}) + f^2_2(\bar{u}),
$$

where $f^2_2(\bar{y}) = 0 \quad \forall \bar{y} \in Y$ and $f^2_2(\bar{u})$ is the support functional to the set $U_{ad}$ at the point $u^0$ (Theorem 10.5 in [20]).

Since all assumptions of the Dubovitskii-Milyutin Theorem are satisfied and we have the suitable adjoint cones, so we are ready to write down the Euler-Lagrange Equation in the following form

$$
f^2_2(\bar{u}) = \lambda_0 \int_Q (\alpha_1 F_{1y} + \alpha_2 F_{2y}) \bar{y} d\tau dt + \lambda_0 \int_{\Sigma} (\alpha_1 F_{1u} + \alpha_2 F_{2u}) \bar{u} d\Gamma dt
$$

(4.21)

\[ \forall (\bar{y}, \bar{u}) \in RTC(Q_2, (y^0, u^0)). \]
where
\[ M^*p(t) = \begin{cases} p(t + \tau_0), & t \in (-\tau_0, T - \tau_0), \\ 0, & t \in (T - \tau_0, T), \end{cases} \]
and taking into account that \( \overline{y} \) is the solution of \( P'(y^0, u^0)(\overline{y}, \overline{u}) = 0 \) for any fixed \( \overline{u} \), we can transform the component with \( \overline{y} \) of the right-hand of (4.21).

In turn, we obtain by using Green’s formula that:
\[
\lambda_0 \int_Q (\alpha_1 F_{1y} + \alpha_2 F_{2y}) \overline{y} dx dt = \lambda_0 \int_{-\tau_0}^T \int_{\mathbb{R}^n} N_1(\alpha_1 F_{1y} + \alpha_2 F_{2y}) \overline{y} dx dt \\
= \lambda_0 \int_{-\tau_0}^T \int_{\Gamma} (\partial p / \partial n + A^* p + M^* p) \overline{y} d\Gamma dt \\
= -\lambda_0 \int_{\Gamma} \partial p / \partial n \overline{y} d\Gamma \bigg|_{-\tau_0}^T + \lambda_0 \int_{-\tau_0}^T \int_{\mathbb{R}^n} p \partial \overline{y} / \partial t d\Gamma dt \\
+ A \overline{y} + M \overline{y} dx dt \\
= \lambda_0 \int_{-\tau_0}^T \int_{\Gamma} \partial p / \partial n \overline{\pi} d\Gamma dt \\
= \lambda_0 \int_\Sigma \partial p / \partial n \overline{\pi} d\Gamma dt,
\]

i.e.,
\[
\lambda_0 \int_Q (\alpha_1 F_{1y} + \alpha_2 F_{2y}) \overline{y} dx dt = \lambda_0 \int_\Sigma \frac{\partial p}{\partial n} \overline{\pi} d\Gamma dt. \tag{4.22}
\]
Substituting (4.22) into (4.21), we get
\[
f_2(\overline{\pi}) = \lambda_0 \int_\Sigma \frac{\partial p}{\partial n} + \alpha_1 F_{1u} + \alpha_2 F_{2u} \overline{\pi} d\Gamma dt \quad \forall \overline{\pi} \in U_{ad}. \tag{4.23}
\]
A number \( \lambda_0 \) in (4.23) cannot be equal to zero, because in this case all functionals in the Euler-Lagrange Equation would also be zero which is impossible according to the Dubovitskii-Milyutin Theorem [34].

Using the definition of the support functional and dividing both sides of the obtained inequalities by \( \lambda_0 \) we finally get the maximum condition (4.15).

If \( RFC(G, (y^0, u^0)) = \emptyset \), then the optimality conditions (4.7)-(4.15) are fulfilled with equality in the maximum condition (4.15). This last remark completes the proof. \( \diamond \)

**Remark 4.1** The results of this paper can easily be generalized to the problem with the performance index in the form of a product of powers of a finite number of integrals.

If \( \alpha_1 = \alpha_2 = 1 \), then the performance index has the form of the product of integrals, while if \( \alpha_1 = 1, \alpha_2 = -1 \), then it becomes the quotient of integrals.

**Remark 4.2** If additionally to the problem (4.1)-(4.5) we assume that
\[ A4) \alpha_1, \alpha_2 \geq 0, \quad \alpha_1^2 + \alpha_2^2 > 0, \]
\[ A5) F_i(x, t, y, u), i = 1, 2 \text{ are strictly convex with respect to the pair } (y, u) \text{ i.e.,} \]
\[
F_i(x, t, \lambda y_1 + (1 - \lambda) y_2, \lambda u_1 + (1 - \lambda) u_2) < \lambda F_i(x, t, y_1, u_1) + (1 - \lambda) F_i(x, t, y_2, u_2),
\]
\[
\forall y_1, y_2, u_1, u_2 \in \mathbb{R}^1, (y_1, u_1) \neq (y_2, u_2), \lambda \in (0, 1).
\]

Then there is the unique solution to the problem (4.1)-(4.5) and the conditions given in Theorem (4.1) are also sufficient for the optimality of the pair \( (y^0, u^0) \). The existence of a unique optimal control \( u^0 \) to the problem under consideration follows from the Weierstrass theorem (Theorem 6.1.4 in [35]). Actually, the set \( U_{ad} \) is weakly sequentially compact (Corollary 6.1.9 in [35]), while the functional \( G(u) = G(y[u], u) \) is strictly convex (it follows from the assumptions (A4),(A5) and the linearity of the state equation) and is weakly lower semi-continuous on \( U_{ad} \) (Theorem 6.1.6 in [35]; \( G(u) \) is continuous since it is the Fréchet differentiable.)

The optimal control \( u^0 \) corresponds with the optimal state \( y^0 \) determined uniquely by the state equation. So the solution to the problem (4.1)-(4.5) exists and it is unique and can given by the pair \( (y^0, u^0) \).

V. **Examples**

Here we shall study some illustrative examples of control problems to which Theorem (4.1) may be applied.
A. Example 1

Let us provide an example that includes a precisely defined set of admissible controls (see Ref. [30]). Let

\[ U_{ad} = \{ v | v \in L^2(\Sigma), v \geq 0 \text{ almost everywhere in } \Sigma \}. \]

Then the optimal control is obtained by the solution of the problem (4.7)-(4.9) and (4.11)-(4.14) but the boundary condition (4.10) replaced by

\[ D^w y(x, t) \geq 0, \quad x \in \Gamma, \quad t \in (-\tau_0, T), \]

and the maximum condition (4.15) is replaced by:

\[ u \geq 0, \left( \frac{\partial p}{\partial n} + \alpha_1 F_{1u} + \lambda \alpha_2 F_{2u} \right) \geq 0, \quad u \left( \frac{\partial p}{\partial n} + \alpha_1 F_{1u} + \lambda \alpha_2 F_{2u} \right) = 0 \text{ on } \Sigma. \]

B. Example 2

We can consider the following boundary Neumann optimization problem:

\[ \frac{\partial y}{\partial t} + Ay + y(t - \tau_0) = 0, \quad x \in \mathbb{R}^n, \quad t \in (0, T), \]
\[ y(x, t) = g(x, t), \quad x \in \mathbb{R}^n, \quad t \in (-\tau_0, 0), \]
\[ y(x, 0) = y_p(x), \quad x \in \mathbb{R}^n, \]
\[ \frac{\partial^w}{\partial A y(x, t)} = u, \quad x \in \Gamma, \quad t \in (-\tau_0, T), \]

where \( \frac{\partial}{\partial n} \) is the co-normal derivatives with respect to \( A(t) \), i.e. \( \frac{\partial^w}{\partial A y} = \frac{\partial}{\partial \nu} \cos(\nu; x_k); \cos(\nu; x_k) = k - \text{th direction cosine of } \nu; \nu \text{ being the normal to the boundary } \Gamma \text{ of } \mathbb{R}^n \text{ for } |\omega| = 0, 1, 2, \ldots, |\omega| \leq \alpha - 1. \)

Let us denote by \( U := L^2(0, T; L^2(\Gamma)) = L^2(\Sigma) \), the space of controls and by \( Y := L^2(-\tau_0, T; W^\infty(\mathbb{R}^n)) \) the space of states. The control time \( T \) is fixed in our problem. The performance functional is given by (4.5), the control constraints is given by (4.6), and we can prove a similar theorem to theorem (4.1) with different boundary conditions i.e., the equation (4.10), is replaced by

\[ \frac{\partial^w}{\partial A y(x, t)} = u^0, \quad x \in \Gamma, \quad t \in (-\tau_0, T), \]

and (4.14) is replaced by

\[ \frac{\partial^w}{\partial A y(x, t)} = 0, \quad x \in \Gamma, \quad t \in (-\tau_0, T). \]

Also we can extend the study to the following mixed distributed Dirichlet-Neumann optimization problem:

\[ \frac{\partial y}{\partial t} + Ay + y(t - \tau_0) = u, \quad x \in \mathbb{R}^n, \quad t \in (0, T), \]
\[ y(x, t) = g(x, t), \quad x \in \mathbb{R}^n, \quad t \in (-\tau_0, 0), \]
\[ y(x, 0) = y_p(x), \quad x \in \mathbb{R}^n, \]
\[ \frac{\partial^w}{\partial A y(x, t)} = 0, \quad x \in \Gamma, \quad t \in (-\tau_0, T), \]
\[ D^w y(x, t) = 0, \quad x \in \Gamma, \quad t \in (-\tau_0, T). \]

C. Example 3 (see [30])

We shall use the following notation:

\[ Q = Q_T = \Omega \times [0, T], \Omega \text{ an open subset of } \mathbb{R}^n; \]
\[ \Sigma = \Sigma_T = \Gamma \times [0, T], \]
\[ \Gamma = \text{boundary of } \Omega, \]
\[ \Sigma = \text{ lateral boundary of } Q. \]

Let \( a_{ij} \) be given function in \( \Omega \times [0, T] = Q \) with \( a_{ij} \in L^\infty(Q) \),

\[ \sum_{i,j=1}^n a_{ij}(x,t)\xi_i\xi_j \geq \mu(\xi_1^2 + \ldots + \xi_n^2), \quad \mu > 0, \xi_i \in \mathbb{R}, \text{ almost every where in } \Omega. \]
Let us take

\[ V = H_0^1(\Omega), \quad A(t)y = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}(a_{ij}(x,t) \frac{\partial y}{\partial x_j}). \]

Let us first consider evolution equations with time delay with the control absent. A typical problem is the following. Let \( \omega > 0 \) be given; we seek \( y(t) \) satisfying

\[ y \in L^2(0,T;\mathbb{V}), y' \in L^2(0,T;\mathbb{V}') \text{ that is } y \in W(0,T), \]

\[ y'(t) + A(t)y(t) + y(t - \omega) = f_1(t), \quad t > \omega, \]

\[ y(t) = g(t) \quad \text{in } ]0,\omega[. \]

with \( f_1 \) given in \( L^2(\omega, T; \mathbb{V}') \) and \( g \) given in \( W(0,\omega) \).

Let us formulate this problem in more convenient function space setting. We introduce the operator \( M \in \mathcal{L}(L^2(0,T;\mathbb{V});L^2(0,T;\mathbb{V}')) \)(for example) by

\[ M_y(t) = \begin{cases} y(t-\omega) & \text{if } t > \omega, \\ 0 & \text{if } t < \omega. \end{cases} \]

Let us define \( f \) and \( y_0 \) by

\[ f(t) = \begin{cases} f_1(t) & \text{if } t > \omega, \\ g' + Ag & \text{if } t < \omega, \end{cases} \quad \text{hence } f \in L^2(0,T;\mathbb{V}), \]

and

\[ y_0 = g(0) \quad \text{(hence } y_0 \in H = L^2(\Omega)). \]

We may replace (assuming (3.6) and (3.7) hold)(4.25) by,

\[ y' + Ay + My = f_1, \quad \text{in } ]0,T[, \]

\[ y(0) = y_0. \]

Indeed, in \( ]\omega,T[ \), (2.29) reduces to the first equation of (2.25) and in \( ]0,\omega[, \) (2.29) becomes

\[ y' + Ay = g' + Ag, \quad y(0) = g(0) \]

and from the uniqueness in Theorem 1.2 [30], we have \( y = g \) in \( ]0,\omega[ \).

The extension of the problem of evolution with delay (4.29) to more general situations is immediate. Let \( t \rightarrow \omega(t) \) be a bounded measurable function which is positive in \( [0,T] \).

For \( y \in W(0,T) \), we define

\[ M_y(t) = y(t - \omega(t)) \quad \text{if } t - \omega(t) \geq 0, 0 \quad \text{if } t - \omega(t) < 0. \]

We then look for \( y \in W(0,T) \), which is a solution of (4.29), \( M \) being given by (4.31). It may be shown [30] that if (3.6), (3.7) as well as (4.30) hold and if \( M \) is given by (4.31), then problem (4.29) admits a unique solution.

**Control Problem.** The state of the system \( y(v) \) is given by

\[ y'(v) + A(t)y(v) + My(v) = f + v, \]

\[ y(0;v) = y_0. \]

Assume that the cost function is given by

\[ J(v) = \int_0^T \| y(t;v) - z_d \|^2 dt + (Nv,v)_\mathbb{V}. \]

Let \( U_{ad} \) be a closed, convex subset of \( U = L^2(Q) \) and let \( u \in U_{ad} \) be the optimal control. We introduce the adjoint state in the following manner:

We note that

\[ M \in \mathcal{L}(C^0([0,T];H);L^2(0,T;H)), \]

and hence

\[ M \in \mathcal{L}(W(0,T);L^2(0,T;H)). \]

We then introduce the adjoint \( M^* \) such that

\[ M^* \in \mathcal{L}(L^2(0,T;H);W(0,T)'). \]
and thereby define the adjoint state $p(u)$ by

$$
- \frac{dp(u)}{dt} + A^*(t)p(u) + M^* p(u) = y(u) - z_d,
$$

(4.36)

The optimal control $u$ is characterized by

$$
\int_0^T (y(t; u) - z_d, y(t; v) - y(t; u)) dt + (Nu, v - u)_U \geq 0 \quad \forall v \in U_{ad}.
$$

Utilizing (4.36), we get,

$$
(p(u) + Nu, v - u)_U \geq 0 \quad \forall v \in U_{ad}.
$$

(4.37)

Hence the optimal control $u$ is determined by the simultaneous solution of (4.32) (with $v = u$), (4.36) and (4.37).

For the case where $\omega(t) = \omega$, we obtain

$$
\begin{align*}
&y'(u) + A(t)y(u) + My(u) = 0, \quad \text{in } Q \\
&y(0; u) = y_0, \\
&y(x, t; u) = u + f_1 \quad \text{on } \Sigma, \\
&-p'(u) + A^*(t)p(u) + M^* p(u) = y(t; u) - z_d, \quad 0 < t < T - \omega, \\
&-p'(u) + A^*(t)p(u) = y(t; u) - z_d, \quad T - \omega < t < T, \\
&p(T; u) = 0,
\end{align*}
$$

to which we must adjoin (4.37).

Finally we can study the following boundary optimization problem:

$$
\begin{align*}
&y'(u) + A(t)y(u) + My(u) = 0, \quad \text{in } Q \\
&y(0; u) = y_0, \\
&y(x, t; u) = u + f_1 \quad \text{on } \Sigma, \\
&-p'(u) + A^*(t)p(u) + M^* p(u) = 0, \quad \text{in } Q \\
&p(T; u) = y(u) - z_d \quad \text{on } \Sigma.
\end{align*}
$$

D. Comments

The main result of the paper contains necessary and sufficient conditions of optimality (of Pontryagin’s type) for infinite order parabolic system that give characterization of optimal control. But it is easily seen that obtaining analytical formulas for optimal control is very difficult. This results from the fact that state equations (4.7)-(4.10), adjoint equation (4.11)-(4.14) and maximum conditions (4.15) are mutually connected that cause that the usage of derived conditions is difficult. Therefore we must resign from the exact determining of the optimal control and therefore we are forced to use approximations methods. Those problems need further investigations and form tasks for future research.

Also it is evident that by modifying:
- the boundary conditions,
- the nature of the control (distributed, boundary),
- the nature of the observation,
- the initial differential system,

an infinity of variations on the above problem are possible to study with the help of Dubovitskii-Milyutin formalism.
REFERENCES