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# On a Nonlinear Delay Difference Equation with Periodic Coefficients

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***Abstract*—In this paper, some results on the existence of positive periodic solutions, the permanence of solutions, and the oscillation of the positive solutions about positive periodic solution of nonlinear delay difference equation with periodic coerfficients**

$$x_{n+1} = \lambda_n x_n + \alpha_n F(x_{n-\omega}), \quad n = 0, 1, \cdots$$

**are obtained.**

**Keywords:** *Nonlinear difference equation, positive periodic, permanence, oscillation*

## I. Introduction

Difference equations have numerous applications in various fields including mathematical biology, see e.g. [1-11] and further references therein. In [1-2], we obtained some results for the asymptotic behaviour of solutions of nonlinear difference equations with time-invariant delay of the form

$$x_{n+1} = \lambda x_n + F(x_{n-m}), \quad n = 0, 1, 2, \cdots$$

where $F : [0, \infty) \to [0, \infty)$ is a continuous function, and $m \geq 0$ is a fixed integer, $x_{-m}, x_{-m+1}, \ldots, x_0$ are the positive initial values and $\lambda \in (0, 1)$ is a given parameter. We applied these results to determine the extinction, persistence, global stability and periodicity conditions in some models of pupulation growth. In [11], E.Braverman and S.H.Saker studied the existence of a positive periodic solution and the attractivity of this solution of the nonlinear delay equation with periodic coerfficients

$$p_{n+1} - p_n = -\delta_n p_n + \frac{\beta_n}{1 + (p_{n-\omega})^m}.$$

S.H.Saker [10] considerd the discrete nonlinear delay survival red blood cell model

$$x_{n+1} - x_n = -\delta_n x_n + p_n e^{-q_n x_{n-\omega}}, \quad n = 1, 2, \cdots$$

where $\{\delta_n\}, \{p_n\}, \{q_n\}$ are positive periodic sequences of period $\omega$ and proved that the equation has a positive periodic solution and estabished some sufficient conditions for oscillations and global attractivity.

Motivated by the work above, in this paper, we consider the nonlinear delay equation with periodic coerfficients

$$x_{n+1} = \lambda_n x_n + \alpha_n F(x_{n-\omega}), \quad n = 0, 1, \cdots \tag{1.1}$$

where $\omega > 0$ is a fixed integer, $\{\lambda_n\}, \{\alpha_n\}$ are positive periodic sequences of priod $\omega$ and $\lambda_n \in (0, 1) \quad \forall n \in \mathcal{N}$,

$$F : [0, \infty) \to (0, \infty) \tag{1.2}$$

and

$$x_{-\omega}, x_{-\omega+1}, \ldots, x_{-1} \in [0, \infty), x_0 > 0 \tag{1.3}$$

are the initial values. First, we apply a cone fixed point theorem due to Krasnosel'skii to prove that the equation has a positive periodic solution $\{x_n^*\}$. Second, we prove that the solutions are permanent and establish some sufficient conditions for oscillation of the positive solutions about $\{x_n^*\}$.

In this paper, we denote by $\mathcal{N}$ as the nonnegative integers numbers, $\mathcal{R}$ as the set of real numbers, $I_\omega = \{0, 1, \cdots, \omega - 1\}$, $\lambda_* = \min_{n \in I_\omega} \lambda_n$, $\lambda^* = \max_{n \in I_\omega} \lambda_n$, $\alpha_* = \min_{n \in I_\omega} \alpha_n$, $\alpha^* = \max_{n \in I_\omega} \alpha_n$.

By a solution of (1.1) we mean a sequences $\{x_n\}$ which is defined for $n \geq -\omega$ and satisfies (1.1) for $n \geq 0$. Then, it is easy to see that the initial value problem (1.1) and (1.3) has a unique positive solution $\{x_n\}$.

A solution $\{x_n\}$ of (1.1) is said to be periodic of prime priod $\omega$, if $\omega$ is the least positive integer for which $x_{n+\omega} = x_n$ for $n = 0, 1, 2, \cdots$.

A solution $\{x_n\}$ of (1.1) is said to be permanent if there exist positive constans $m$ and $M$ with $0 < m \leq M < \infty$ such that for any initial conditions satisfies (1.3) there exists a positive integer $n_1$ which depends on the initial conditions such that

$$m \leq x_n \leq M \quad n \geq n_1.$$

A solution $\{x_n\}$ of (1.1) is said to oscillate about the sequence $\{x_n^*\}$ if the terms $x_n - x_n^*$ of the sequence $\{x_n - x_n^*\}$ are neither eventually positive nor eventually negative.

## II. The results

**Theorem A** (Krasnosel'skii fixed point theorem) ([3]). *Assume that $X$ is a Banach space, and $E$ is a cone in $X$, $\Omega_1$ and $\Omega_2$ are open subsets of $X$ such that $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$. Suppose that*

$$T : E \cap \left(\overline{\Omega_2} \backslash \Omega_1\right) \to E$$

*is a completely continuous operator and satisfied either*

*$||Tx|| \geq ||x||$ for any $x \in E \cap \partial\Omega_1$ and $||Tx|| \leq ||x||$ for any $x \in E \cap \partial\Omega_2$*

*or*

*$||Tx|| \leq ||x||$ for any $x \in E \cap \partial\Omega_1$ and $||Tx|| \geq ||x||$ for any $x \in E \cap \partial\Omega_2$.*

*Then, $T$ has a fixed point in $E \cap \left(\overline{\Omega_2} \backslash \Omega_1\right)$.*

**Theorem B** ([9]) *Let $k$ be a positive integer and let $\{p_n\}$ be a sequence of nonnegative real numbers such that*

$$\sum_{j=0}^{k-1} p_{n+j} > 0$$

*for $n > 0$. Assume that $\{x_n\}$ is a solution of inequality*

$$x_{n+1} - x_n + p_n x_{n-k} \leq 0$$

*such that*

$$x_n > 0 \quad n \geq -k.$$

*Then equation*

$$A_{n+1} - A_n + p_n A_{n-k} = 0$$

*has a solution $\{A_n\}$ such that*

$$0 < A_n \leq x_n \text{ for } n \geq -k \text{ and } \lim_{n\to\infty} A_n = 0.$$

Let $X = \{x = \{x_n\} : x_{n+\omega} = x_n, \quad n \in \mathcal{N}\}$, $||x|| = \max\{|x_n| : n \in \mathcal{N}\}$, $E = \{x \in X : x_n \geq \prod_{r=0}^{\omega-1} \lambda_r ||x||\}$. Then $X$ is a Banach space endowed with the norm $||.||$ and $E$ is a cone.

Define a mapping $T : X \to X$ by

$$\{Tx\}_n = \sum_{s=0}^{\omega-1} H(n,s)\alpha_s F(x_{s-\omega}), x \in X,$$

where

$$H(n,s) = \frac{\prod_{r=s+n+1}^{n+\omega-1} \lambda_r}{1 - \prod_{r=0}^{\omega-1} \lambda_r}.$$

Clearly $T$ is a completely continuous operator on $X$. It can be proved that a periodic solution of (1.1) is equivalent to establishing a fixed point of operator $T$.

Firstly, we have the following lemmas.

**Lemma 2.1.** *Assume that (1.2)-(1.3) are satisfied. Then $\{x_n\}$ is a positive $\omega-$periodic solution of (1.1) if and only if $\{x_n\}$ is a positive $\omega-$periodic solution of difference equation*

$$x_n = \sum_{s=0}^{\omega-1} H(n,s)\alpha_s F(x_{s-\omega}). \tag{1.4}$$

**Lemma 2.2.** *Assume that (1.2)-(1.3) are satisfied. If $\{x_n\}$ is a positive $\omega-$periodic solution of (1.1), then*

$$x_n \geq \prod_{r=0}^{\omega-1} \lambda_r ||x||, n \in \mathcal{N}.$$

**Lemma 2.3.** *Assume that (1.2)-(1.3) are satisfied. Then*

$$TE \subset E.$$

The proofs of these lemmas are simple, so we omit them here.

Secondly, we study the existence of positive bounded periodic solutions of (1.1). We have the following theorem.

**Theorem 2.4.** *Assume that (1.2)-(1.3) and the following conditions are satisfied:*

*$H_1$. There exists $L > 0$ such that $F(t) > \frac{L\left(1 - \prod_{s=0}^{\omega-1} \lambda_s\right)}{\prod_{s=0}^{\omega-1} \lambda_s \omega \alpha_*}$, for $t \in [\prod_{r=0}^{\omega-1} \lambda_r L, L]$.*

*$H_2$. There exists $0 < k < L$ such that $F(t) < \frac{k\left(1 - \prod_{s=0}^{\omega-1} \lambda_s\right)}{\omega \alpha^*}$, for $t \leq k$.*

*Then (1.1) has at least one positive bounded $\omega-$periodic solution.*

*Proof.* Let

$$A = \frac{\prod_{s=0}^{\omega-1} \lambda_s}{1 - \prod_{s=0}^{\omega-1} \lambda_s}, B = \frac{1}{1 - \prod_{s=0}^{\omega-1} \lambda_s}, \sigma = \prod_{r=0}^{\omega-1} \lambda_r,$$

$$\Omega_L = \{x \in X : ||x|| < L\}.$$

Then, for any $x \in E \cap \partial\Omega_L$, we have

$$x_{n-\omega} \geq \sigma ||x|| = \sigma L.$$

This implies $x_{n-\omega} \in [\sigma L, L]$. Hence,

$$\{Tx\}_n \geq A \sum_{s=0}^{\omega-1} \alpha_s F(x_{s-\omega}) > A \sum_{s=0}^{\omega-1} \frac{L}{A\omega} = A\omega \frac{L}{A\omega} = ||x||.$$

This yields $||Tx|| > ||x||, \forall x \in E \cap \partial\Omega_L$.

Let $\Omega_k = \{x \in X : ||x|| < k\}$. Then, for any $x \in E \cap \partial\Omega_k$, we have

$$\{Tx\}_n \leq B \sum_{s=0}^{\omega-1} \alpha_s F(x_{s-\omega}) < B \sum_{s=0}^{\omega-1} \frac{k}{B\omega} \leq B\omega \frac{k}{B\omega} = ||x||.$$

It implies that $||Tx|| < ||x||, \forall x \in E \cap \partial\Omega_k$. Thus $T$ satisfies all the requirements in Theorem A. Therefore $T$ has a fixed point in $E \cap \left(\overline{\Omega_L} \backslash \Omega_k\right)$, denote $\{x_n^*\}$.

We have $\{Tx^*\}_n = x_n^* = \sum_{s=0}^{\omega-1} H(n,s)\alpha_s F(x_{s-\omega})$, $k < ||x^*|| < L$ and $x_n^* \geq \sigma ||x^*|| \geq \sigma k > 0$, which shows that $\{x_n^*\}$ is a positive $\omega-$periodic bounded solution of (1.4). By Lemma 2.1, $\{x_n^*\}$ is a positive bounded $\omega-$periodic solution of difference equation (1.1). The proof is complete.

Next, we establish a condition for every solution of (1.1) is permanent.

**Theorem 2.5.** *Assume that (1.2)-(1.3) are satisfied, $F$ is decreasing and $B = F(0)$. Then every solution $\{x_n\}_{n=-\omega}^{\infty}$ of (1.1) is permanent and*

$$\limsup_{n\to\infty} x_n \leq \frac{\alpha^* B}{1 - \lambda^*}.$$

*Proof.* We have

$$x_1 = \lambda_0 x_0 + \alpha_0 F(x_{-\omega}) > 0,$$

which proves that $x_n > 0 \quad (n = 1, 2, \cdots)$ by induction. Let

$$M = \max\{x_{-\omega}, x_{-\omega+1}, \cdots, x_0, \frac{\alpha^* B}{1-\lambda^*}\}.$$

From (1.1), we see that

$$\begin{aligned} x_1 &= \lambda_0 x_0 + \alpha_0 F(x_{-\omega}) \\ &\leq \lambda^* M + \alpha^* B \leq \lambda^* M + (1-\lambda^*)M = M. \end{aligned}$$

We prove that $x_n \leq M$ for all $n$. Indeed assuming by induction that $x_k \leq M$ for all $k \leq n$. Then by the difference equation (1.1)

$$\begin{aligned} x_{n+1} &= \lambda_n x_n + \alpha_n F(x_{n-\omega}) \\ &\leq \lambda^* M + \alpha^* B \\ &\leq \lambda^* M + (1-\lambda^*)M = M. \end{aligned}$$

Therefore $x_n \leq M$ for all $n$. On the other hand from (1.1 ), we have

$$x_{n+1} = \lambda_n x_n + \alpha_n F(x_{n-\omega}) \geq \alpha_* F(M).$$

Hence,

$$\alpha_* F(M) \leq x_n \leq M, \quad n \geq 1,$$

i.e., every solution of (1.1) is positive and permanent.

Define a sequence $\{y_n\}$, by

$$y_{n+1} = \lambda^* y_n + \alpha^* B, \quad y_0 = x_0.$$

It is easy to see that,

$$x_n \leq y_n = (\lambda^*)^n x_0 + \alpha^* B \frac{1-(\lambda^*)^n}{1-\lambda^*}.$$

Letting $n \to \infty$, we have

$$\limsup_{n\to\infty} x_n \leq \frac{\alpha^* B}{1-\lambda^*}.$$

Thus, the proof of Theorem 2.5 is complete.

Finally, we find conditions that every positive solution of (1.1) oscillates about positive periodic solution.

**Theorem 2.6.** *Assume that (1.2)-(1.3), $(H_1)$, and $(H_2)$ are satisfied and $F$ is decreasing. Then every nonoscillatory solution $\{x_n\}$ of (1.1) satisfies*

$$\lim_{n\to\infty} (x_n - x_n^*) = 0.$$

*Proof.* Let $\{x_n\}$ be a nonoscillatory solution of (1.1) about $\{x_n^*\}$. Then there exists a sufficiently large integer $n_1 > 0$ such that $x_n > x_n^*$ or $x_n < x_n^*$ for $n \geq n_1$. Assume that $x_n > x_n^*$ for $n \geq n_1$. The proof when $x_n < x_n^*$ for $n \geq n_1$ is similar and will be omitted. Put

$$z_n = x_n - x_n^*.$$

Then $z_n > 0$ and satisfies the difference equation

$$z_{n+1} - \lambda_n z_n + \alpha_n[F(x_{n-\omega}^*) - F(z_{n-\omega} + x_{n-\omega}^*)] = 0.$$

Since $F$ is decreasing, we have

$$z_{n+1} + (\lambda_n - 1)z_n \leq 0 \quad \forall n \geq n_1. \tag{1.5}$$

It implies

$$z_{n+1} \leq (1-\lambda_n)z_n < z_n.$$

Hence, $\{z_n\}$ is decreasing and there exists a nonnegative real number $\alpha \geq 0$ such that

$$\lim_{n\to\infty} z_n = \alpha.$$

Assume by contradiction, we prove that $\alpha = 0$. Otherwise, assume that $\alpha > 0$, then there exists a positive integer $n_2 > n_1$ such that $\frac{2\alpha}{3} \leq z_n \leq \frac{4\alpha}{3}$ for $n > n_2$. From (1.5) we have

$$z_{n+1} - z_n \leq -\lambda_n z_n \leq -\frac{2\alpha}{3}\lambda_n \leq -\frac{2\alpha}{3}\lambda_* \tag{1.6}$$

for $n > n_2$. Summing up both sides of (1.6) from $n_2$ to $n-1$, we obtain

$$z_n \leq z_{n_2} - \frac{2\alpha}{3}\lambda_*(n - n_2) \to -\infty$$

as $n \to \infty$. This is a contradiction. Then $\lim_{n\to\infty} z_n = 0$. The proof is complete.

**Theorem 2.7.** *Assume that (1.2)-(1.3), $(H_1)$, and $(H_2)$ are satisfied, $F$ is differentiable and decreasing. Then every solution $\{x_n\}$ of (1.1) oscillates about $\{x_n^*\}$ if every solution of the linear equation*

$$y_{n+1} - y_n + (\epsilon-1)\alpha_n F'(\theta_n)\Big(\prod_{1=n-\omega}^{n} \lambda_i\Big)^{-1} y_{n-\omega} = 0 \tag{1.7}$$

*oscillates (where $\epsilon \in (0,1)$, and $\{\theta_n\}_n$ is generated by Lagrange Theorem).*

*Proof.* Assume that (1.1) has a nonoscillatory solution $\{x_n\}$. Then there exists a sufficiently large integer $n_1 > 0$ such that $x_n > x_n^*$ or $x_n < x_n^*$ for $n \geq n_1$. Assume that $x_n > x_n^*$ for $n \geq n_1$. The proof when $x_n < x_n^*$ for $n \geq n_1$ is similar and will be omitted. Put $z_n = x_n - x_n^*$. Then $z_n > 0$ and satisfies the difference equation

$$z_{n+1} - \lambda_n z_n + \alpha_n[F(x_{n-\omega}^*) - F(z_{n-\omega} + x_{n-\omega}^*)] = 0. \tag{1.8}$$

By Lagrange Theorem, (1.8) can be rewritten as

$$z_{n+1} - \lambda_n z_n - \alpha_n F'(\theta_n) z_{n-\omega} = 0, \tag{1.9}$$

where $\theta_n \in (x_{n-\omega}^*, z_{n-\omega} + x_{n-\omega}^*)$. It follows that for any given arbitrarily $\epsilon > 0$ we have

$$z_{n+1} - \lambda_n z_n + (\epsilon-1)\alpha_n F'(\theta_n) z_{n-\omega} \leq 0. \tag{1.10}$$

Set $z_n = y_n \prod_{1=0}^{n-1} \lambda_i$, then $y_n > 0$ and satisfies the difference inequality

$$y_{n+1} - y_n + (\epsilon-1)\alpha_n F'(\theta_n)\Big(\prod_{1=n-\omega}^{n} \lambda_i\Big)^{-1} y_{n-\omega} \leq 0.$$

But by Theorem B (1.9) has an eventually positive solution, which contradicts the assumption that every solution of (1.7 ) oscillates. There fore every positive solution of (1.1) oscillates about $\{x_n^*\}$. The proof is complete.

By the oscillation results due to Erbe and Zhang [5], G. Ladas, Ch.G. Philos and Y.G. Sficas [9], we see that every solution of (1.7) oscillates if one of the following conditions is satisfies

$$\liminf_{n\to\infty} (\epsilon-1)\alpha_n F'(\theta_n)\Big(\prod_{1=n-\omega}^{n} \lambda_i\Big)^{-1} > \frac{\omega^\omega}{(\omega+1)^{\omega+1}}, \tag{1.11}$$

$$\limsup_{n\to\infty} \sum_{s=n-\omega}^{n} (\epsilon-1)\alpha_s F'(\theta_s)\Big(\prod_{\imath=s-\omega}^{s} \lambda_i\Big)^{-1} > 1, \quad (1.12)$$

$$\liminf_{n\to\infty} \frac{1}{\omega} \sum_{s=n-\omega}^{n-1} (\epsilon-1)\alpha_s F'(\theta_s)\Big(\prod_{\imath=s-\omega}^{s} \lambda_i\Big)^{-1} > \frac{\omega^\omega}{(\omega+1)^{\omega+1}}. \quad (1.13)$$

**Theorem 2.8.** *Assume that the assumptions of Theorem 2.7 are satisfied. If one of inequalities (1.11), (1.12) and (1.13) holds, then every positive solution of (1.1) oscillates about positive periodic solution* $\{x_n^*\}$.

## III. Conclusion

In this paper, we show a condition for the existence of positive bounded periodic solutions of (1.1) (Theorem 2.4). We obtain a condition that all solutions of (1.1) are permanent (Theorem 2.5), some conditions that every positive solution of (1.1) oscillates about positive bounded periodic solution (Theorem 2.6, Theorem 2.7, Theorem 2.8).

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